Smoothing using Geodesic Averages

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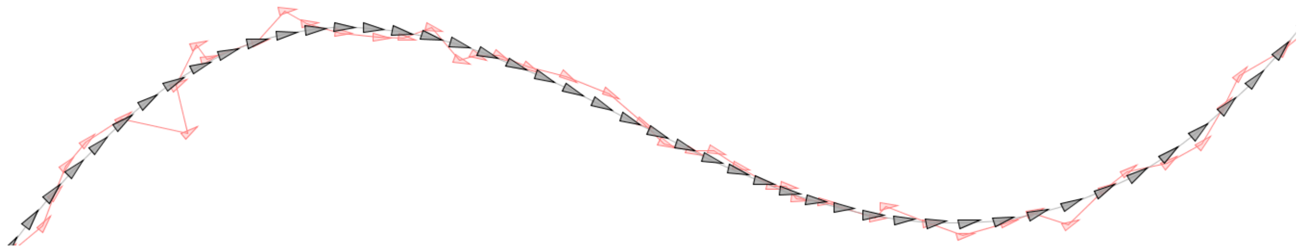


Figure: A sequence of noisy localization estimates of a fast driving go-kart in red, and the result of smoothing in black. ■

Abstract: Geodesic averages have been used to generalize curve subdivision and Bézier curves to Riemannian manifolds and Lie groups. We show that geodesic averages are suitable to perform smoothing of sequences of data in nonlinear spaces. In applications that produce temporal uniformly sampled manifold data, the smoothing removes high-frequency components from the signal. As a consequence, discrete differences computed from the smoothed sequence are more regular. Our method is therefore a simpler alternative to the extended Kalman filter. We apply the smoothing technique to noisy localization estimates of mobile robots. ■

Keywords: smoothing, nonlinear filter, geodesics, special Euclidean group, curve subdivision, mobile robots

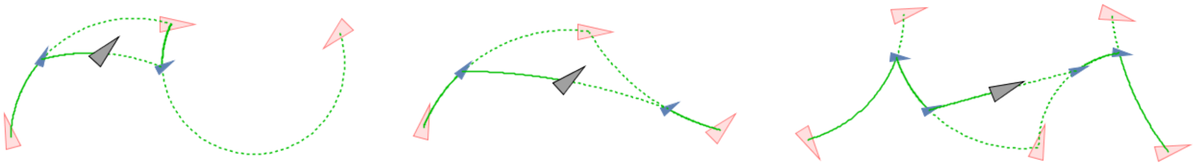


Figure: Geodesic averages in the Lie group $\overline{\text{SE}}(2)$ used in applications: Left, quartic B-spline refinement used in curve subdivision. Center, mean of a 3-point sequence based on Dirichlet weights. Right, mean of a 5-point sequence based on Gaussian weights used for smoothing. The input points are indicated as red arrowheads. The geodesic average is indicated in gray. ■

Introduction

During a certain period in history, points in the Euclidean plane were constructed using ruler and compass only. Given the coordinates (0, 0) and (1, 0) to start with, the limited set of operations yields numbers such as the square root of two, but not π.

The construction of binary averages along geodesics in a nonlinear space M is of equal puritan spirit: The single permitted operation to produce a new point in M is to evaluate the geodesic τ that connects two sufficiently close points p and q in M at a certain ratio $\lambda \in \mathbb{R}$. For the ratios $\lambda = 0$, and $\lambda = 1$, the geodesic is defined to give $\tau(0) = p$, and $\tau(1) = q$. The operation is denoted with $[p, q]_\lambda$. The points p and q are drawn from an initial collection of points in M.

Example: In the Euclidean space, geodesics are straight lines, and $[p, q]_\lambda = p + \lambda(q - p)$ is the affine combination between two points $p, q \in \mathbb{R}^n$. ■

Geodesic averages of the binary type $[p, q]_\lambda$, or in nested fashion, for instance $[[p, q]_\sigma, r]_\lambda$ appear in the literature. However, there is no unified terminology for the expressions. [2007 Wallner/Yazdani/Grohs] use the term “geodesic combinations”. [2014 Dyn/Sharon] refer to “geodesic average between two points on the manifold”, “geodesic weighted average”, “repeated binary average”, and “geodesic inductive mean”.

Definition: The term *binary average* refers to an expression of the form $[p, q]_\lambda$. The term *geodesic average* refers to the nested computation of binary averages, for instance $[[p, q]_\sigma, r]_\lambda$ for $p, q, r \in M$ and $\lambda, \sigma \in \mathbb{R}$. ■

In the article, we use geodesic averages to perform smoothing of a sequence of points from a Riemannian manifold, or Lie group. The geodesic averages are derived from window functions that are commonly used in the convolution of linear signals to attenuate high-frequency components. Discrete differences of Lie group-valued sequences show the low-pass filter characteristic of the nonlinear smoothing operator.

The article is structured as follows: We recap the concept of geodesics in Riemannian manifolds and Lie groups. We present previous work on curve subdivision and Bézier curves in nonlinear spaces in a unified framework. Then, we define smoothing using geodesic averages. The new smoothing method is applied to sequences of manifold-valued data generated by real-world mobile robots. In the conclusion, we list further applications of geodesic averages.

Geodesics in Riemannian Manifolds and Lie Groups

Let M be a Riemannian manifold. Between two sufficiently close points $p, q \in M$ with $p \neq q$ a unique shortest path $\tau_{p,q} : \mathbb{R} \to M$ exists with parameterization proportional to arc-length and $\tau_{p,q}(0) = p$, and $\tau_{p,q}(1) = q$. We refer to $\tau_{p,q}$ as the *geodesic* that connects p and q. For all $p \in M$, we define $\tau_{p,p}(\lambda) := p$ for all $\lambda \in \mathbb{R}$. For enhanced readability, when referring to $\tau_{p,q}$, we assume that the points $p, q \in M$ are sufficiently close so that the geodesic $\tau_{p,q}$ is well-defined and unique.

The binary average is defined as short-hand $[p, q]_\lambda := \tau_{p,q}(\lambda)$. The shortest path from p to q is the same as the shortest path from q to p reversed: The relation $[p, q]_\lambda = [q, p]_{1-\lambda}$ holds for all $p, q \in M$ and $\lambda \in \mathbb{R}$. In particular, $[p, q]_{1/2} = [q, p]_{1/2}$.

Example: The 2-dimensional sphere $M = S^2$ of unit radius embedded in the 3-dimensional Euclidean space $\mathbb{R}^3$ centered at the origin is a Riemannian manifold. For two points $p = (1, 0, 0)$, and $q = \left(0, 1/\sqrt{5}, 2/\sqrt{5}\right)$ from the sphere, and $\lambda = 0.4$ the binary average is $[p, q]_\lambda \approx (0.8090, 0.2628, 0.5257)$. The general formula for the binary average $[p, q]_\lambda$ for two, non-antipodal points $p, q \in S^2$ is stated in a later section. ■

For a Lie group $(M, .)$ the geodesics are parameterized by

$$[p, q]_\lambda = p.\exp(\lambda \log(p^{-1}.q)) \quad \text{for sufficiently close } p, q \in M \text{ and } \lambda \in \mathbb{R}.$$

The group action "." and the inverse p^{-1} are available for a Lie group but not for a general manifold.

Example: Consider the rotation group $M = SO(3)$. An element $p \in SO(3)$ is a 3×3 orthogonal matrix with determinant +1. The group action is matrix multiplication. The matrix logarithm maps to the vector space of skew-symmetric 3×3 matrices. An implementation of the logarithm is stated in [2009 Chirikjian]. The matrix exponential is obtained using [1815 Rodrigues] formula. ■

Curve Generation using Geodesic Averages

Geodesic averages were proposed for the definition of Bézier curves, and curve subdivision in nonlinear spaces, and analyzed in [1995 Park/Ravani] and [2007 Wallner/Yazdani/Grohs]. We revisit the constructions to gain an intuition for the concept, and facilitate the introduction of smoothing in the next section.

In Euclidean space, the computation of an affine combination of points does not depend on the ordering of the added terms. In contrast, a geodesic average consists of binary averages that do not commute in a nonlinear space. The design requirements for a geodesic average from a given affine combination are

- the geodesic average reduces to the affine combination when evaluated in Euclidean space, and
- if the affine weight mask is symmetric, the structure of the geodesic average should reflect the symmetry.

When a geodesic average involves more than two points, there is not a unique solution to the constraints.

We visualize the nested structure of a geodesic average as a binary tree. A binary average is represented as a non-leaf node. The geodesic split ratio determines the position of such a node for better intuition. If you try and take a cat apart to see how it works, the first thing you have on your hands is a non-working cat.

Curve Subdivision

[2007 Wallner/Yazdani/Grohs] analyze convergence and smoothness of curve subdivision based on geodesic averages. The authors show that "the nonlinear schemes essentially have the same properties regarding C1 and C2 smoothness as the linear schemes they are derived from". The article features an illustration of curve subdivision in SE(3).

[2014 Dyn/Sharon] derive conditions on curve subdivision which "guarantee convergence from any initial manifold-valued sequence. The definition and analysis of convergence are intrinsic to the manifold." The convergence analysis is carried out for several schemes including cubic and quartic B-spline refinement. The authors point out that the framework does not yield a contractivity factor for the quintic B-spline scheme.

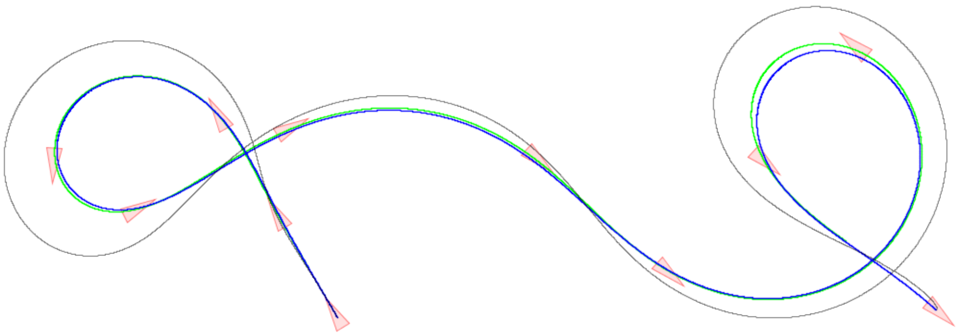


Figure: Comparison of two limit curves generated by subdivision in $\overline{SE}(2)$. The curve in green is based on quartic B-spline refinement. The curve in blue is based on quintic B-spline rules with curvature comb indicated in gray. The eleven control points that define the curves are indicated as red arrowheads. ■

[2018 Hakenberg] uses curve subdivision in $\overline{SE}(2)$ for the intuitive design of planar curves with favorable curvature that are suitable for use as trajectories for car-like mobile robots.

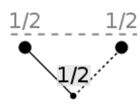
In the following, M denotes a Riemannian manifold or Lie group. We reproduce the generalization of cubic and quartic B-spline refinement to geodesic averages from previous work.

Cubic B-spline subdivision requires midpoint insertion and vertex repositioning. For points $p, q, r \in \mathbb{R}^n$ in the Euclidean space, the operations are the affine combinations:

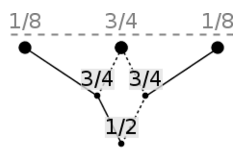
midpoint insertion	$\frac{1}{2} p + \frac{1}{2} q$
vertex repositioning	$\frac{1}{8} p + \frac{3}{4} q + \frac{1}{8} r.$

For the ordered 3-point sequence $p, q, r \in M$ the geodesic averages are

midpoint insertion $[p, q]_{1/2}$



vertex repositioning $[[p, q]_{3/4}, [r, q]_{3/4}]_{1/2}$



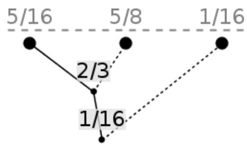
Another possibility for vertex repositioning is the geodesic average $[[p, r]_{1/2}, q]_{3/4}$.

Quartic B-spline subdivision is a dual scheme and requires only the definition of a single affine combination. For points $p, q, r \in \mathbb{R}^n$ in Euclidean space the computation is

vertex insertion $\frac{5}{16} p + \frac{5}{8} q + \frac{1}{16} r.$

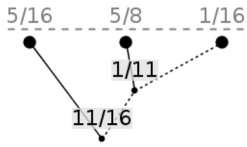
For the ordered sequence of three points $p, q, r \in M$, [2014 Dyn/Sharon] suggest to use the combination of two geodesics for which the authors ascertain contractivity

vertex insertion $[[p, q]_{2/3}, r]_{1/16}$



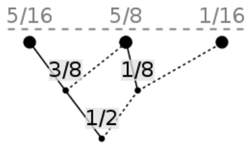
Another variant that uses only two geodesics defines

vertex insertion $[p, [q, r]_{1/11}]_{11/16}$



[2018 Hakenberg] proposes the averaging along three geodesics

vertex insertion $[[p, q]_{3/8}, [q, r]_{1/8}]_{1/2}$



Initial experiments have not indicated a qualitative advantage of one variant over another. The limit curves are very similar. In fact, there are infinitely many combinations of using three binary averages that all simplify to the weight mask [5/16, 5/8, 1/16] in the Euclidean space:

vertex insertion $[[p, q]_\alpha, [q, r]_\beta]_\gamma$ for $\alpha = \frac{16\gamma - 11}{16(\gamma - 1)}$, $\beta = \frac{1}{16\gamma}$, and $\gamma \in (0, 1) \subset \mathbb{R}$.

The three cases listed above correspond to $\gamma = 1/16$, $\gamma = 11/16$, and $\gamma = 1/2$ respectively. ■

The discourse shows that an affine combination in the Euclidean space does not translate to a unique geodesic average in general. Different nested binary averages exist that satisfy the design requirements. Because geodesic averages do not commute in nonlinear spaces, the choice of the geodesic average impacts the result of curve subdivision. Reality is frequently inaccurate.

Bézier Curves

[1995 Park/Ravani] define Bézier curves on Riemannian manifolds. For a moment, nothing happened. Then, after a second or so, nothing continued to happen. [2007 Popiel/Noakes] join Bézier curves on a manifold so that the resulting curve is C2 at the intersection point. The article features the hyperbolic 2-space as an example.

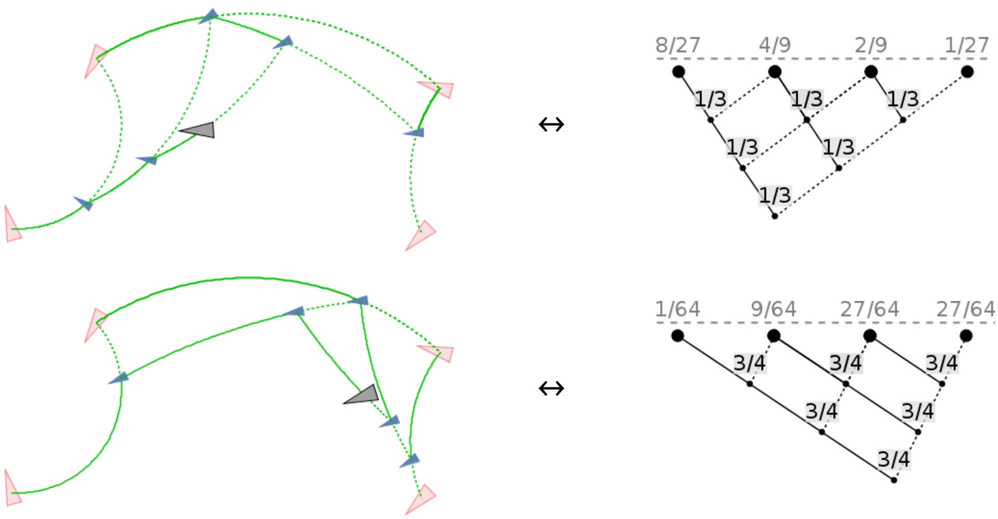


Figure: Geodesic average that evaluates a Bézier curve spanned by four control points in $\overline{\text{SE}}(2)$ at parameter values 1/3 and 3/4 respectively, and the corresponding binary tree that encodes the nesting of the binary averages. ■

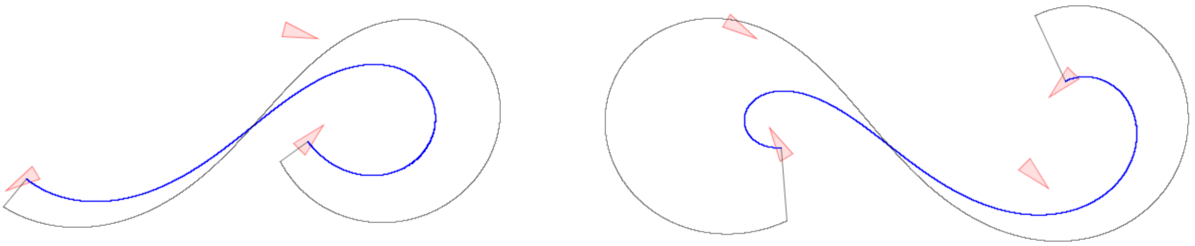


Figure: Two Bézier curves generated by three and four control points in $\overline{\text{SE}}(2)$ respectively projected to the plane in blue. The curvature comb is indicated in gray. ■

Smoothing

In Euclidean space, the smoothing of a sequence of data is the convolution of the sequence with a kernel of affine weights. Every point in the sequence is replaced with the affine combination evaluated from the range-n neighborhood of the original point.

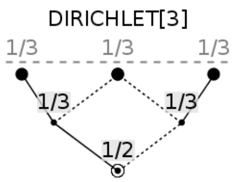
For the smoothing of a sequence of values from a Riemannian manifold, or Lie group M, we propose a similar procedure. However, a geodesic average takes the place of the affine combination: Every point in the sequence is replaced with the geodesic average evaluated from the range-n neighborhood of the original point.

Moving average “filters data by replacing every value by the mean value in its range-n neighborhood” [Mathematica, MeanFilter]. For $n = 1$, the mean of three points is computed as

replacement of q $\quad \frac{1}{3}p + \frac{1}{3}q + \frac{1}{3}r.$

For the ordered sequence of three points $p, q, r \in M$ we define the geodesic average as

replacement of q $\quad [[p, q]_{1/3}, [r, q]_{1/3}]_{1/2}$

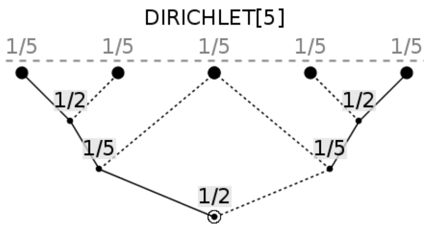


For a range-2 neighborhood, the mean of five points is computed as

replacement of r $\quad \frac{1}{5}p + \frac{1}{5}q + \frac{1}{5}r + \frac{1}{5}s + \frac{1}{5}t.$

For the ordered sequence of five points $p, q, r, s, t \in M$ we define the geodesic average as

replacement of r $\quad [[[p, q]_{1/2}, r]_{1/5}, [[t, s]_{1/2}, r]_{1/5}]_{1/2}$



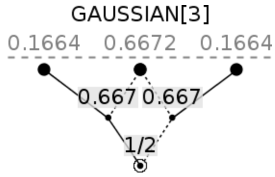
Moving average is an example of a linear smoothing filter, in which the affine weights in the mask used in the convolution are all equal. In general, the weights in the mask are sampled from a window function, and normalized to add up to 1. The window function determines the spectral properties of the linear filter. Examples of window functions are Blackman, Gaussian, Hamming, Hann, Nuttall, Parzen, and Tukey, see [2018 Wikipedia].

Example: The Gaussian window function w is defined as

$w : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ $\quad$ with $\quad w(x) = \exp(-\frac{50}{9}x^2).$

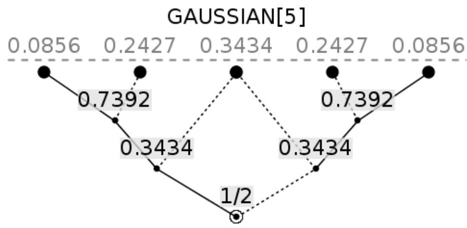
The smoothing mask of length 3 is the evaluation of the window function at parameter values $x \in \{-1/2,\ 0,\ 1/2\}$, with the resulting values normalized to add up to 1. We obtain $m \approx [0.1664, 0.6672, 0.1664]$.

For the ordered sequence of three points $p, q, r \in M$ we define the geodesic average as replacement of q $\quad [[p, q]_{0.6672}, [r, q]_{0.6672}]_{1/2}$



For the ordered sequence of five points $p, q, r, s, t \in M$ we define the geodesic average as

replacement of r $\quad [[[p, q]_\alpha, r]_\beta, [[t, s]_\alpha, r]_\beta]_{1/2}$



with $\alpha \approx 0.7392$ and $\beta \approx 0.3434$. In the diagrams, the decimal values are rounded. ■

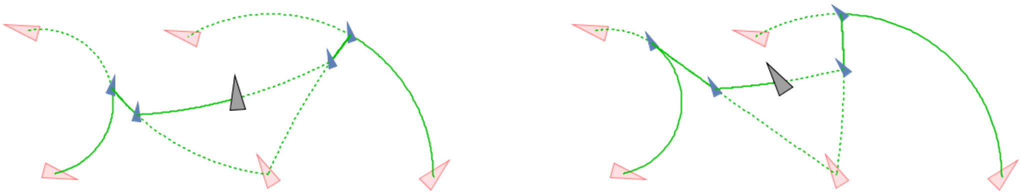


Figure: Averaging a sequence of 5 points in $\overline{\text{SE}}(2)$ using a geodesic average defined by Dirichlet (left), and Gaussian (right) window function. ■

A symmetric affine weight mask of width 3 has the form $[\omega_1, 1 - 2\,\omega_1, \omega_1]$. To evaluate the center of a sequence of 3 points $p, q, r \in M$, the geodesic average of the form $[[p, q]_\alpha, [r, q]_\alpha]_{1/2}$ is used. We require that the geodesic average reduces to the affine combination in Euclidean space. Thus, $\alpha = 1 - 2\,\omega_1$.

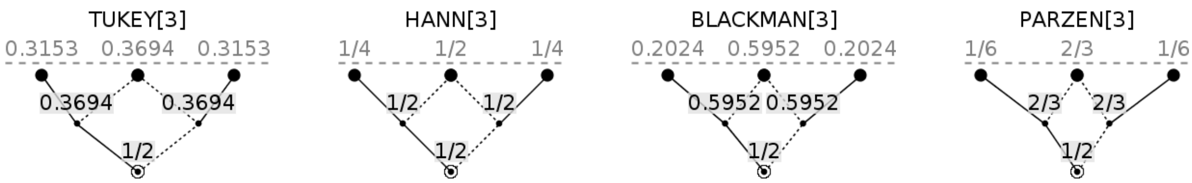


Figure: Binary trees that encode geodesic averages of the form $[[p, q]_\alpha, [r, q]_\alpha]_{1/2}$ with weights derived from common window functions. ■

A symmetric affine weight of width 5 has the form $[\omega_1, \omega_2, 1 - 2\,\omega_1 - 2\,\omega_2, \omega_2, \omega_1]$. To evaluate the center of a sequence of 5 points $p, q, r, s, t \in M$, the geodesic average of structure $[[[p, q]_\alpha, r]_\beta, [[t, s]_\alpha, r]_\beta]_{1/2}$ is used. The requirement that the geodesic average reduces to the affine combination in Euclidean space implies $\beta = 1 - 2\,\omega_1 - 2\,\omega_2$, and $\alpha = \omega_2 / (\omega_1 + \omega_2)$.

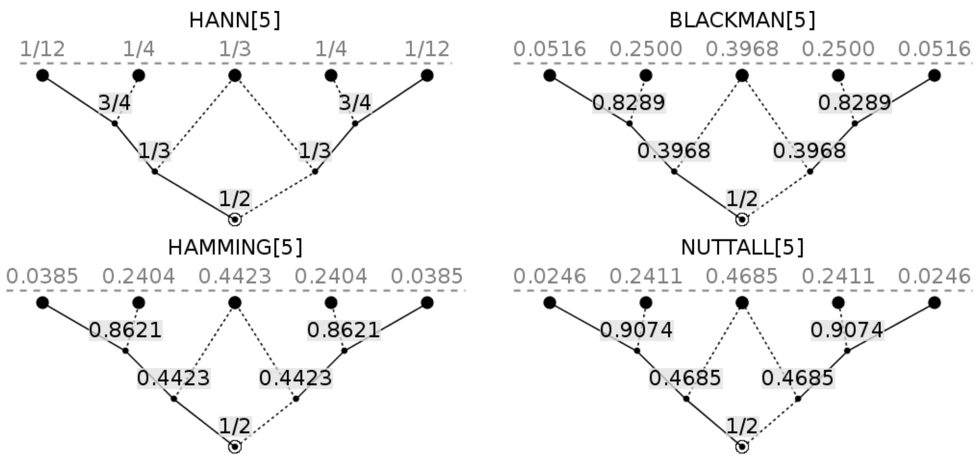


Figure: Binary trees that encode geodesic averages of the form $[[[p, q]_\alpha, r]_\beta, [[t, s]_\alpha, r]_\beta]_{1/2}$ with weights derived from common window functions. ■

The implementation [2018 IDSC-Frazzoli] solves the split ratios in the system of nonlinear equations given symmetric affine masks of arbitrary size. We refrain from stating the general expressions here.

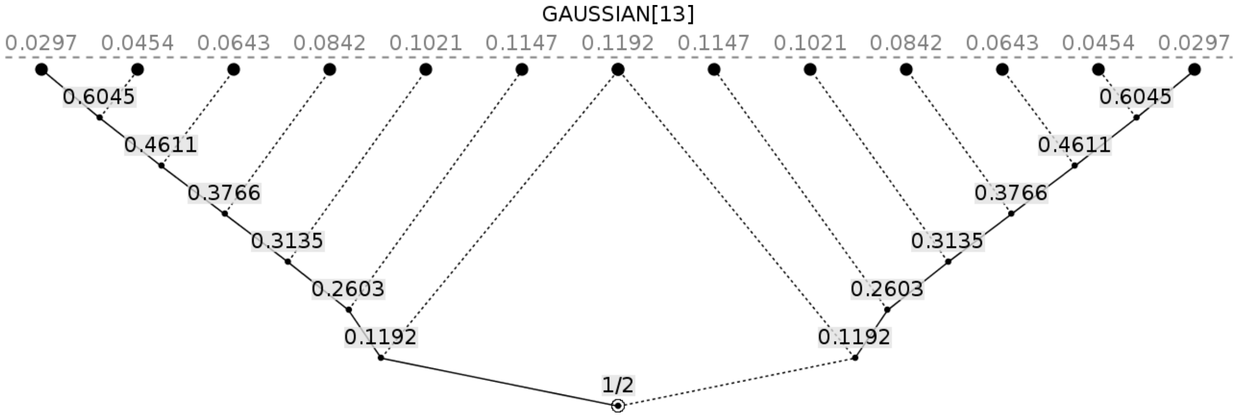


Figure: Binary tree with split ratios derived from a symmetric affine mask of width 13 sampled from the Gaussian window function. ■

In Euclidean space the impact of a filter is quantified in the frequency domain before and after smoothing. Frequency decomposition of a manifold-valued sequence is not available in general. However, for a Lie group-valued sequence, frequency analysis of the sequence of discrete differences between all successive points from the original sequence is available. The discrete difference between two points $p, q \in M$ from a Lie group is the expression $\log(p^{-1}.q)$, which is a value in the Lie algebra, and vector space. That means the sequence of discrete differences permits coordinate-wise discrete frequency decomposition.

The previous paragraph motivates the use of established window functions to derive geodesic averages for use in smoothing operators.

A smoothing filter in Euclidean space reproduces signals that are uniformly sampled from a constant function, or a linear function. Smoothing based on geodesic averages reproduces sequences that are uniformly sampled from a single geodesic in a Riemannian manifold, or Lie group.

Applications of Smoothing using Geodesic Averages

We apply the smoothing method based on geodesic averages to sequences of pose data in the special Euclidean group SE(2) ,and SE(3). In each case, the data originates from a real mobile robot. Then, we demonstrate the smoothing of a sequence of synthetic data with random perturbations in S^2.

Our implementation of geodesic averages in SE(2) is based on [2018 Hakenberg]. The implementation for SE(3) makes use of the formulas for $\exp : \mathfrak{se}(3) \to \mathrm{SE}(3)$ and $\log : \mathrm{SE}(3) \to \mathfrak{se}(3)$ stated in [2017 Ethan Eade]. There is no point in using the word ‘impossible’ to describe something that has clearly happened.

Smoothing the Pose of a Go-Kart in SE(2)

We apply smoothing to the pose sequence produced from a lidar-based localization estimation of a go-kart in motion. The pose of the vehicle is represented by an element in the Lie group SE(2). The sampling rate is 20[Hz]. The vehicle was driven around several laps of a race track. The repository [2018 IDSC-Frazzoli] contains a reference to the dataset.

For smoothing, we use a geodesic average derived from the Gaussian window function of width 13 corresponding to a duration of 0.65[s] in real-time. The discrete difference between two points $p, q \in \mathrm{SE}(2)$ with sampling time difference Δt is a value in the Lie algebra $\mathfrak{se}(2)$, and computed as $(\Delta t)^{-1} \log(p^{-1}.q)$.

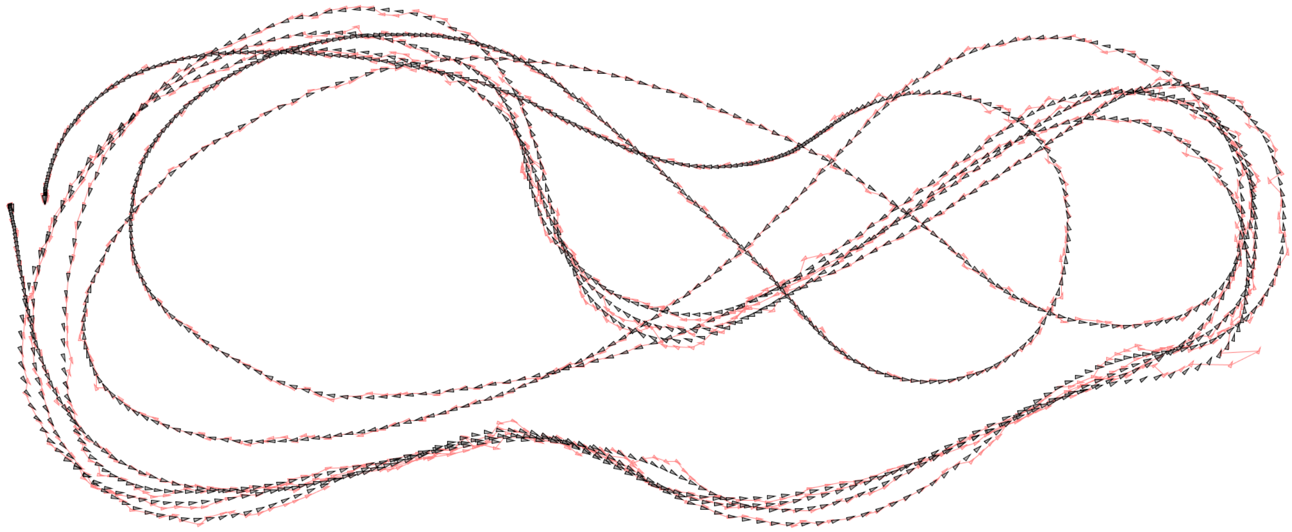


Figure: Sequence of vehicle pose estimates of a fast driving go-kart in red. The result of applying the smoothing filter is indicated in gray. ■

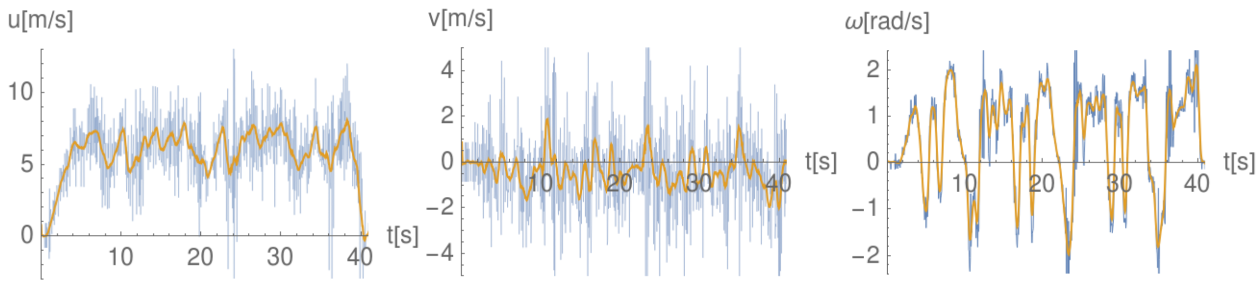


Figure: Discrete differences yield the tangent velocity u, side-slip velocity v, and heading rate ω before (blue) and after smoothing (yellow).

The sequence of discrete differences evaluated from the *raw* pose sequence exhibits high-frequency components. The sequence of discrete differences evaluated from the *smoothed* pose sequence is physically more plausible. The conventional estimation of these derivatives requires the design of an extended Kalman filter, and typically incorporates additional sensor information, see [2008 Pierre Pettersson].

Smoothing the Pose of a Micro Aerial Vehicle in SE(3)

We apply smoothing to the sequence of poses in SE(3) defined in the dataset 'MH_04_difficult' from [2016 Burri et al.]. The position and orientation estimate of the aerial vehicle is sampled at a rate of 200[Hz]. We use a geodesic average derived from the Gaussian window function of width 49 corresponding to a duration of 0.245[s] in real-time.

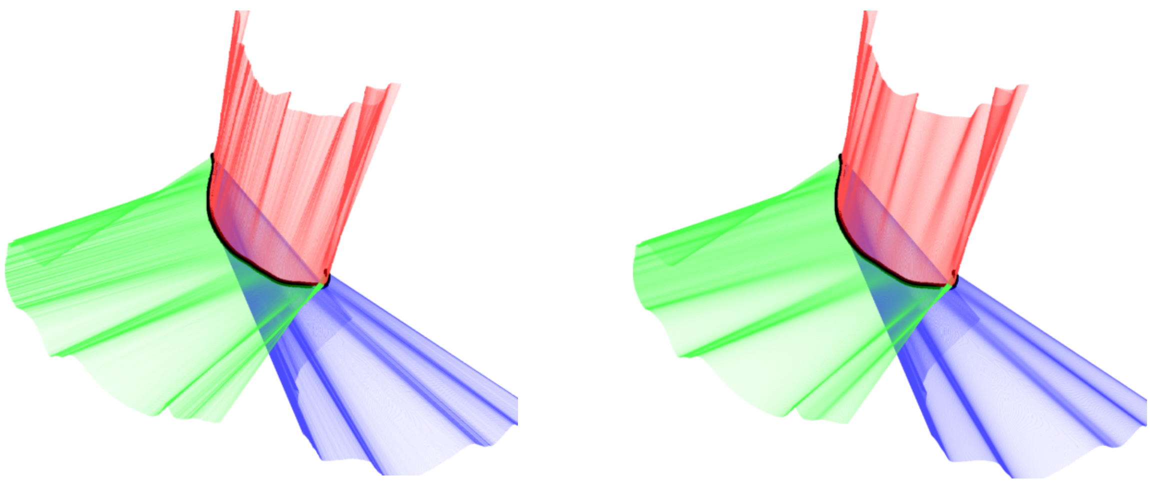


Figure: Raw pose in black, and the 3-axis orientation frame in red, green, blue of the dataset in the time interval 52.5 – 60.5[s] left, as well as the corresponding part of the smoothed sequence, right. ■

The advantage of smoothing the raw sequence in SE(3) is evident when computing discrete differences. The numerical derivative between two points p, $q \in$ SE(3) with sampling time difference Δt is a value in the Lie algebra $\mathfrak{se}(3)$, and computed as $(\Delta t)^{-1} \log(p^{-1}.q)$.

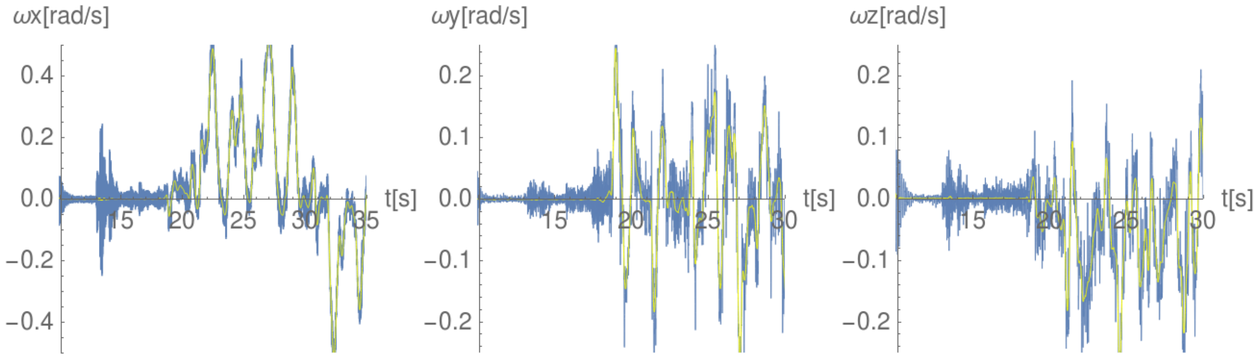


Figure: The orientation components of the discrete differences before (blue) and after the smoothing (yellow) during the time 10 – 35[s]. ■

The smoothing operation attenuates high-frequency components in the sequence. During the time 10 – 18[s] the drone is stationary and does not rotate at all. This fact is better reflected in the smoothed sequence.

2-Dimensional Sphere

The 2-dimensional sphere $M = S^2$ of unit radius embedded in Euclidean space centered at the origin with induced metric is a Riemannian manifold. Any geodesic on S^2 is a great circle. The geodesic that connects two non-antipodal points p, $q \in S^2$ is parameterized by $\lambda \in \mathbb{R}$ using the formula for $[p, q]_\lambda$ as

```
SO3ad[{ωx_, ωy_, ωz_}] := ( 0   -ωz   ωy
                            ωz   0   -ωx
                           -ωy   ωx    0 )
```

```
S2Split[p_, q_, λ_] := With[{a = ArcCos[p.q]},
  If[Sin[a] == 0, p, MatrixExp[SO3ad[Cross[p, q] λ a / Sin[a]]].p]]
```

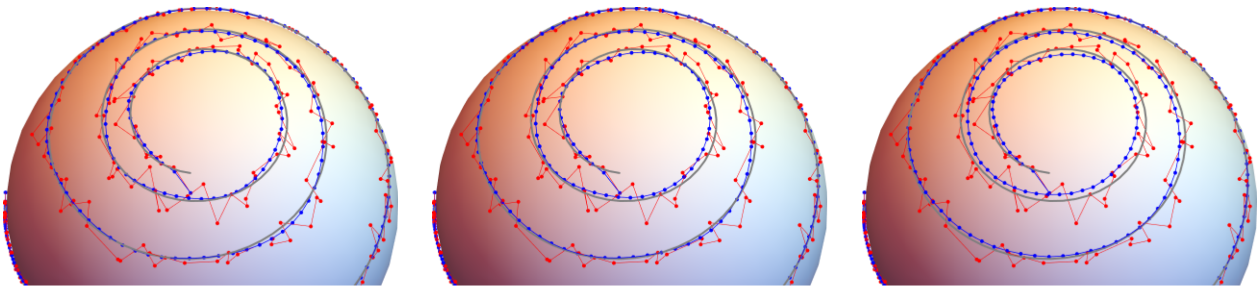


Figure: Sequence of perturbed points (red) from a loxodrome (gray) are smoothed using a mask of width 15 with weights derived from the Blackman, Hann, and Gaussian window functions. The result is the sequence in blue. ■

Conclusion

We propose the smoothing of sequences of points from a Riemannian manifold, or Lie group using geodesic averages. We motivate the design of geodesic averages based on conventional window functions. The geodesic average substitutes the affine combination in conventional convolution. Besides that, the method is parameter free.

Experiments on datasets from actual mobile robots in SE(2) and SE(3) indicate that the smoothed pose sequence is physically more plausible than the raw data. Evidence are the more regular discrete differences of the smoothed sequence.

Despite the nonlinear terms, we found the smoothing method presented in the article to be robust in practical applications. The method is simpler than alternatives such as the bidirectional extended Kalman filter, see [1965 Rauch/Tung/Striebel].

Future Work

We plan to investigate the following extensions:

• The smoothing method discussed in the article is symmetric around a sample and repositions the data in an offline fashion. An *online estimation* is an affine combination between the past estimate and the current measurement, which can be formulated as a geodesic average as well.

• Geodesic averages can be adapted to the smoothing of sequences of *non-uniformly* sampled data.

• The geodesic average corresponding to a given affine combination is not unique. Therefore, *guidelines* for the choice of geodesic average backed by analytical, or experimental results are useful.

• Feature preserving filters exist for linear spaces, see [2011 Gastal/Oliveira]. For feature preserving filters, the weight mask depends on the input data. For instance, the geodesic average may depend on the distance of data points.

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