Remapping and KE conservation

Consider a single component of horizontal velocity (the other component can be treated the same way). Denote the data before remapping as $u_1^-, \ldots, u_N^-$ and the data after remapping as $u_1^+, \ldots, u_N^+$. The original layer thicknesses are $h_1^-, \ldots, h_N^-$ and the updated ones are $h_1^+, \ldots, h_N^+$. (Consider the layer thicknesses to be the ones interpolated from h points on the C grid to the velocity points as appropriate for whatever component of velocity we're dealing with.)

Define

$$\mathrm{ke}^\pm = \frac{1}{2}\sum_n h_n^\pm (u_n^\pm)^2.$$

KE non-conservation occurs when $\mathrm{ke}^- \neq \mathrm{ke}^+$.

First split the velocity into barotropic and baroclinic components where the barotropic component is defined to be

$$u_t^\pm = \frac{1}{H}\sum_n h_n^\pm u_n^\pm$$

(where H is the total depth) and the baroclinic component is defined to be

$$u_n^{\pm,c} = u_n^\pm - u_t^\pm.$$

(The t subscript means baroTropic while the c means baroClinic.) Notice that the depth integrated baroclinic part is zero (obviously)

$$\sum_n h_n^\pm u_n^{\pm,c} = \sum_n h_n^\pm (u_n^\pm - u_t^\pm) = (\sum_n h_n^\pm u_n^\pm) - H u_t^\pm = 0.$$

Second assume that the remap exactly preserves the barotropic component of the velocity.

Third notice that the total KE is the sum of the KE in the barotropic and baroclinic components:

$$\mathrm{ke}^\pm = \frac{1}{2}\sum_n h_n^\pm (u_n^\pm)^2 = \frac{1}{2}\sum_n h_n^\pm (u_t + u_n^{\pm,c})^2 = \mathrm{ke}_t^\pm + u_t \sum_n h_n^\pm u_n^{\pm,c} + \mathrm{ke}_c^\pm = \mathrm{ke}_t^\pm + \mathrm{ke}_c^\pm.$$

Conclude that the remap exactly conserves the barotropic component of the kinetic energy. Energy non-conservation is associated with errors in the baroclinic part of the velocity only.

How to conserve KE We want to adjust the baroclinic part of the remapped velocity so that it has the right KE. First compute ke_c^- then simply update the remapped baroclinic component so that it has the right KE:

$$u_n^{+,c} = \left[\frac{\mathrm{ke}_c^-}{\mathrm{ke}_c^+}\right]^{1/2} \tilde{u}_n^{+,c}$$

where the tilde means that

> $\tilde{u}_n^{+,c}$ **is the baroclinic part after remapping but before being corrected to conserve KE.**

Note that this *can* change the sign of the velocity. For example, you could start with a velocity profile that is entirely non-negative, and end up with one that has small negative values. For velocity this doesn't really matter (in my opinion), but it does mean that this correction shouldn't be applied to things that really need to remain positive, like concentrations.

Error analysis The foregoing update conserves KE (*and also conserves momentum*), but does it mean that the updated velocity profile is no longer accurate? Suppose that we have remapped using an order-p scheme, i.e.

$$u_n^{\text{true}} = \tilde{u}_n^+ + e_n$$

where the error is

$$e_n = \mathcal{O}(h^p).$$

If we make an adjustment of order h^p to the remapped velocity, then it will still be accurate to order h^p. The un-corrected KE is

$$\mathrm{ke}^+ = \frac{1}{2} h_n^+ \left(u_n^{\mathrm{true}} - e_n\right)^2 = \mathrm{ke}^+_{\mathrm{true}} + \mathcal{O}(h^p) = \mathrm{ke}^- + \mathcal{O}(h^p).$$

where we assumed that $\mathrm{ke}^+_{\mathrm{true}} = \mathrm{ke}^-$. Now subtract off the barotropic component, which we know to be correct, so

$$\mathrm{ke}_c^+ = \mathrm{ke}_c^- + \mathcal{O}(h^p).$$

Now divide and take the square root to get

$$\left[\frac{\mathrm{ke}_c^-}{\mathrm{ke}_c^+}\right]^{1/2} = \sqrt{1 - \mathcal{O}(h^p)} = 1 - \frac{1}{2}\mathcal{O}(h^p).$$

This shows that we are making an order h^p adjustment to the remapped velocity. If you start with something that is order h^p accurate, and then add something to it that is order h^p, the result is still order h^p accurate. So the KE-conserving adjustment does not change the order of the underlying remapping scheme.

Caveat and Threshold Sometimes the remapping scheme is simply not accurate. A statement like 'the scheme is order p' means that as the grid is refined the error *eventually* decreases at a particular rate. For a fixed grid and a fixed true velocity, the error may be large. In such cases it is possible for the correction factor

$$\left[\frac{\mathrm{ke}_c^-}{\mathrm{ke}_c^+}\right]^{1/2}$$

to be far from 1, when the error analysis says it should be close to 1. In such cases we want to limit the amount of correction that we make to avoid (e.g.) significantly increasing the magnitude of the baroclinic velocity. The correction scheme is therefore limited as follows

$$u_n^{+,c} = \min\left\{1.25, \left[\frac{\mathrm{ke}_c^-}{\mathrm{ke}_c^+}\right]^{1/2}\right\} \tilde{u}_n^{+,c}.$$

So the correction can *reduce* the baroclinic velocity to 0, but it can't *amplify* it more than 25%. The 25% threshold is arbitrary but seems to be enough to prevent instability.