Generalize ablastr::coarsen::average for any staggering combination

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1 Definitions

Define the following variables

- The staggering is a bit such that $s = 0$ for a cell-centered field, and $s = 1$ for a nodal field.
- Let s_{CrseDes} and s_{FineSrc} denote the staggerings for the coarse destination and fine source fields, respectively.
- Let r denote the coarsening ratio. We will assume it is an integer satisfying $r \geq 1$.
- Let h_{CrseDes} and h_{FineSrc} be the cell sizes for their respective grids.
- Let indices such as i, j, k be for the coarse destination array, and indices such as $\text{ii}, \text{jj}, \text{kk}$ be for the fine source array.
- Let $\phi(x)$ be the field of interest, and let CrseDes_i and $\text{FineSrc}_{\text{ii}}$ be the arrays.
- Let $w_{i,\text{ii}}$ be the weight on the contribution to the coarse destination at index i from the fine source at index ii. We have

  $$\text{CrseDes}_i = \sum_{\text{ii}=\text{ii}_{\text{min},i}}^{\text{ii}_{\text{max}.i}} w_{i,\text{ii}} \text{FineSrc}_i \tag{1}$$

  where $[\text{ii}_{\text{min},i}, \text{ii}_{\text{max},i}]$ is the range of non-zero weights, and $n = \text{ii}_{\text{max},i} - \text{ii}_{\text{min},i} + 1$ is the number of non-zero weights.

2 Find source fine index ranges

Let $\langle \text{ii} \rangle_i$ be the midpoint of the fine source such that

$$\text{CrseDes}_i \equiv \text{FineSrc}_{\langle \text{ii} \rangle_i} \tag{2}$$

Note that $\langle \text{ii} \rangle_i$ may not necessarily be an integer. Let us solve for $\langle \text{ii} \rangle_i$ in terms of the staggerings.

The coarse destination and fine source arrays are related to the field by

$$\text{CrseDes}_i = \phi\left(x_{\text{low}} + (i + \frac{1 - s_{\text{CrseDes}}}{2})h_{\text{CrseDes}}\right) \tag{3}$$

$$\text{FineSrc}_{\text{ii}} = \phi\left(x_{\text{low}} + (\text{ii} + \frac{1 - s_{\text{FineSrc}}}{2})h_{\text{FineSrc}}\right) \tag{4}$$

Note that the coarse and fine cell sizes are related by

$$h_{\text{CrseDes}} = r h_{\text{FineSrc}} \tag{5}$$

Thus we have

$$(i + \frac{1 - s_{\text{CrseDes}}}{2})h_{\text{CrseDes}} = (\langle\text{ii}\rangle_i + \frac{1 - s_{\text{FineSrc}}}{2})h_{\text{FineSrc}} \tag{6}$$

$$\Rightarrow \qquad \langle\text{ii}\rangle_i = ri + r\frac{1 - s_{\text{CrseDes}}}{2} - \frac{1 - s_{\text{FineSrc}}}{2} \tag{7}$$

Define the cutoff points that bound the midpoint of the fine sources as

$$\begin{aligned} \text{ii}^{(\pm)}_{\text{cutoff},i} &= \frac{1}{2}\left(\langle\text{ii}\rangle_i + \langle\text{ii}\rangle_{i\pm1}\right) & (8)\\ &= \langle\text{ii}\rangle_i \pm \frac{1}{2}r & (9)\\ &= ri + r\left(\frac{1 - s_{\text{CrseDes}}}{2} \pm \frac{1}{2}\right) - \frac{1 - s_{\text{FineSrc}}}{2} & (10)\\ &= ri + \frac{-rs_{\text{CrseDes}} + s_{\text{FineSrc}} - 1}{2} + r\frac{1 \pm 1}{2} & (11) \end{aligned}$$

These are integers if $(s_{\text{FineSrc}} - rs_{\text{CrseDes}} - 1)$ is even. The range is then

$$\boxed{\text{ii}_{\text{min},i} = \left\lceil \text{ii}^{(-)}_{\text{cutoff},i} \right\rceil = ri + \left\lceil \frac{s_{\text{FineSrc}} - rs_{\text{CrseDes}} - 1}{2} \right\rceil} \tag{12}$$

$$\boxed{\text{ii}_{\text{max},i} = \left\lfloor \text{ii}^{(+)}_{\text{cutoff},i} \right\rfloor = ri + \left\lfloor \frac{s_{\text{FineSrc}} - rs_{\text{CrseDes}} - 1}{2} \right\rfloor + r} \tag{13}$$

3 Derive the weights

3.1 Conditions on the weights

The conditions on the weights can be listed as follows:

- Must sum to 1 for each destination index i:

$$\sum_{\text{ii}} w_{i,\text{ii}} = 1 \qquad \forall \qquad i \tag{14}$$

- Charge conservation: the total weight on each source index ii must be equal:

$$\sum_{i} w_{i,\text{ii}} = \frac{1}{r} \qquad \forall \qquad \text{ii} \tag{15}$$

- Symmetric about $\langle \text{ii} \rangle_i$ as there is no preferred direction. This also makes the first order error zero:

$$w_{i,\text{ii}} = w_{i,\text{jj}} \quad \forall \quad \text{ii}, \text{jj} \quad \text{s.t.} \quad \text{ii} + \text{jj} = 2 \langle \text{ii} \rangle_i \tag{16}$$

- Minimize the second order error for each i:

$$\text{minimize} \sum_{\text{ii}} w_{i,\text{ii}} (\text{ii} - \langle \text{ii} \rangle_i)^2 \tag{17}$$

- Non-negativity for the sake of simplicity:

$$w_{i,\text{ii}} \geq 0 \tag{18}$$

which prevents complicated stencils based on canceling higher order errors.

3.2 Results

Given these conditions on the weights, one can show that the weights are given by the following:

If $(s_{\text{FineSrc}} - r s_{\text{CrseDes}} - 1)$ is even, then

$$\text{ii}_{\text{min},i} = ri + \frac{s_{\text{FineSrc}} - r s_{\text{CrseDes}} - 1}{2} \tag{19}$$

$$\text{ii}_{\text{max},i} = ri + \frac{s_{\text{FineSrc}} - r s_{\text{CrseDes}} - 1}{2} + r \tag{20}$$

$$n = r + 1 \tag{21}$$

$$w_{i,\text{ii}} = \begin{cases} 1/r & \text{ii}_{\text{min},i} + 1 \leq \text{ii} \leq \text{ii}_{\text{max},i} - 1 \\ 1/(2r) & \text{ii} = \text{ii}_{\text{min},i} \quad \text{or} \quad \text{ii} = \text{ii}_{\text{max},i} \\ 0 & \text{else} \end{cases} \tag{22}$$

If $(s_{\text{FineSrc}} - r s_{\text{CrseDes}} - 1)$ is odd, then

$$\text{ii}_{\text{min},i} = ri + \frac{s_{\text{FineSrc}} - r s_{\text{CrseDes}}}{2} \tag{23}$$

$$\text{ii}_{\text{max},i} = ri + \frac{s_{\text{FineSrc}} - r s_{\text{CrseDes}}}{2} - 1 + r \tag{24}$$

$$n = r \tag{25}$$

$$w_{i,\text{ii}} = \begin{cases} 1/r & \text{ii}_{\text{min},i} \leq \text{ii} \leq \text{ii}_{\text{max},i} \\ 0 & \text{else} \end{cases} \tag{26}$$

4 Special cases

In the following, let $d = s_{\text{CrseDes}}$, $s = s_{\text{FineSrc}}$, and $I_{\text{min}} = \text{ii}_{\text{min},i}$

4.1 Special case of $r = 1$

If $r = 1$, then in the four staggering cases we have

- $d = 0$ and $s = 0$:
  - $(s - rd - 1) = -1$ is odd
  - $I_{\text{min}} = ri + \frac{s-rd}{2} = \boxed{i}$ and $n = r = \boxed{1}$
- $d = 0$ and $s = 1$
  - $(s - rd - 1) = 0$ is even
  - $I_{\text{min}} = ri + \frac{s-rd-1}{2} = \boxed{i}$ and $n = r + 1 = \boxed{2}$
- $d = 1$ and $s = 0$
  - $(s - rd - 1) = -2$ is even
  - $I_{\text{min}} = ri + \frac{s-rd-1}{2} = \boxed{i - 1}$ and $n = r + 1 = \boxed{2}$
- $d = 1$ and $s = 1$
  - $(s - rd - 1) = -1$ is odd
  - $I_{\text{min}} = ri + \frac{s-rd}{2} = \boxed{i}$ and $n = r = \boxed{1}$

4.2 Special case of $r = 2$

If $r = 2$, then in the four staggering cases we have

- $d = 0$ and $s = 0$:
  - $(s - rd - 1) = -1$ is odd
  - $I_{\text{min}} = ri + \frac{s-rd}{2} = \boxed{2i}$ and $n = r = \boxed{2}$
- $d = 0$ and $s = 1$
  - $(s - rd - 1) = 0$ is even
  - $I_{\text{min}} = ri + \frac{s-rd-1}{2} = \boxed{2i}$ and $n = r + 1 = \boxed{3}$
- $d = 1$ and $s = 0$
  - $(s - rd - 1) = -3$ is odd
  - $I_{\text{min}} = ri + \frac{s-rd}{2} = \boxed{2i - 1}$ and $n = r = \boxed{2}$
- $d = 1$ and $s = 1$
  - $(s - rd - 1) = -2$ is even
  - $I_{\text{min}} = ri + \frac{s-rd-1}{2} = \boxed{2i - 1}$ and $n = r + 1 = \boxed{3}$